STEIN MANIFOLDS AND MULTIPLICITY-FREE REPRESENTATIONS OF COMPACT LIE GROUPS

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1. Introduction

This paper is an outgrowth of a talk, whose aim was to give a survey of recent results on multiplicity-free representations of compact Lie groups arising in complex analysis. Some proofs are given and some results are new. For example, spherical complex spaces are defined with respect to any (and not just compact) real form of a complex reductive algebraic group G. Theorem 4.3 is a new characterization of such spaces in terms of representation theory. A stronger result (Theorem 4.4) was known for compact real forms $K \subset G$. In that case one has the so called complexification theorem which allows to consider Stein spaces acted on by K as K-invariant domains in Stein spaces with G-action. For a non-compact real form we have to make an additional assumption that there exists an open embedding with similar properties.

In the theory of compact complex homogeneous manifolds, a considerable role is played by manifolds fibered by tori over flag manifolds. It turns out that a compact complex homogeneous manifold X of a reductive algebraic group is spherical if and only if X has such a fibration, see Theorem 3.5. This new result shows that non-algebraic spherical complex spaces have interesting geometric properties.

All complex spaces are assumed to be reduced and irreducible. The central question of the paper is the existence of antiholomorphic involutions which have nice behaviour with respect to group actions. Namely, let K be a connected compact Lie group acting by holomorphic transformations on a complex space X. For X spherical and Stein, we want to construct an antiholomorphic involution $\mu : X \to X$ which is θ-equivariant with respect to a Weyl involution $\theta : K \to K$. The reason is that, for holomorphically separable complex spaces, it is then easy to check whether the algebra $\mathcal{O}(X)$ is multiplicity-free as a K-module. This is the case if and only if μ preserves all K-orbits, see Theorem 6.4. Moreover, for any complex space the existence of some antiholomorphic involution preserving each K-orbit is sufficient for $\mathcal{O}(X)$ to be multiplicity-free, see Theorem 5.2.

A θ-equivariant antiholomorphic involution μ is constructed for spherical Stein manifolds, see [A2] and Theorem 6.9 below. The key point is the algebraic part of the theorem, i.e., the similar construction for smooth affine spherical varieties. Any such variety is a fiber bundle over a spherical affine homogeneous variety G/H with a spherical H-module as a fiber. The isotropy subgroup $H \subset G$ is reductive and belongs to the class of so called adapted subgroups, see Sect. 6. We construct μ with required properties for $X = G \times_H V$,

where H is a connected adapted subgroup and V is an arbitrary regular H-module. The latter construction is new, see Theorem 6.7. This theorem does not solve the problem if H is disconnected. The proof in the general case can be found in [A2].

2. Preliminaries

Let K be a compact Lie group. We will use the standard notation $\hat{K}$ for the set of all equivalence classes of irreducible representations of K. For each $\delta \in \hat{K}$ we denote by V_δ an irreducible K-module corresponding to δ. Then, of course, V_δ is a finite-dimensional Hilbert space, on which K acts by a unitary representation. However, we do not specify an invariant scalar product on V_δ, which is unique up to a positive multiple. Let $d(\delta)$ be the dimension of V_δ, ξ_δ the character of V_δ, and $\chi_\delta = d(\delta)\xi_\delta$.

We now recall a fundamental result of Harish-Chandra on the representations of compact Lie groups in Fréchet spaces, see [H]. Let ϱ be a continuous representation of K in a Fréchet space F. Then there is a family of continuous operators

$$E_\delta : F \to F, \quad E_\delta(f) = \int_K \overline{\chi_\delta(x)} \cdot \varrho(x) f \cdot d\mu(x) \quad (f \in F,\ \delta \in \hat{K}),$$

where μ is the normalized Haar measure on K. The orthogonality relations for characters imply that

$$E_\delta^2 = E_\delta, \quad E_\delta \cdot E_\epsilon = 0 \quad (\delta \neq \epsilon),$$

and each E_δ commutes with all $\varrho(k),\ k \in K$. It follows that

$$F_\delta := E_\delta(F) = \{f \in F \,|\, E_\delta f = f\}$$

is a closed K-invariant subspace in F.

Definition 2.1. F_δ is called the isotypic component of F of type δ. A representation ϱ is called multiplicity-free if each non-zero isotypic component is irreducible, i.e., either $F_\delta = \{0\}$ or F_δ is isomorphic to V_δ as a K-module.

A vector $f \in F$ is said to be differentiable if the mapping $K \to F,\quad k \mapsto \varrho(k)f$, is of class C^∞. The subspace of differentiable vectors is denoted by F^∞. A theorem of Gårding says that F^∞ is dense in F. This result is valid for a continuous representation of any Lie group countable at infinity.

A vector $f \in F$ is said to be K-finite if the linear span of the orbit $\varrho(K)f$ is a finite-dimensional subspace. The subspace F^0 of K-finite vectors is contained in F^∞. In our case K is compact and we have $F_\delta \subset F^0$ for all $\delta \in \hat{K}$. Indeed, for $f \in F_\delta$ and $k \in K$ one has

$$\varrho(k)f = \varrho(k)E_\delta f = E_\delta \varrho(k) f = \int_K \overline{\chi_\delta(x)} \cdot \varrho(xk) f \cdot d\mu(x) = \int_K \overline{\chi_\delta(xk^{-1})} \cdot \varrho(x) f \cdot d\mu(x)$$

by the invariance of μ. Now, if $a_{ij}(x)$ are the matrix elements with respect to some basis of V_δ and the vectors $f_{ij} \in F$ are defined by

$$f_{ij} = \int_K \overline{a_{ij}(x)} \cdot \varrho(x) f \cdot d\mu(x)$$

then $\varrho(K)f$ is contained in the linear span of f_{ij}.

Theorem 2.2 (Harish-Chandra, see [H]). *For $f \in F^\infty$ one has*

$$f = \sum_{\delta \in \hat{K}} E_\delta f \ ,$$

where the convergence is absolute with respect to any continuous seminorm on F. The series is finite (has only finitely many non-zero terms) if and only if $f \in F^0$.

This is a central result which has many applications. For example, Theorem 2.2 shows that F^0 is dense in F. Here is another corollary. Recall that the commuting algebra of a representation ϱ is the algebra of all continuous endomorphisms $A : F \to F$ satisfying $A\varrho(k) = \varrho(k)A$ for all $k \in K$.

Theorem 2.3. *A representation ϱ is multiplicity-free if and only if the commuting algebra $\mathcal{A}(\varrho)$ is commutative.*

Proof. An endomorphism from $\mathcal{A}(\varrho)$ preserves the isotypic components. If ϱ is multiplicity-free then, by Schur's lemma, two such endomorphisms commute on each F_δ. By Theorem 2.2 they commute everywhere on F.

Conversely, assume ϱ is not multiplicity-free. Then at least one F_δ contains two distinct irreducible submodules V_1 and V_2. On $V = V_1 + V_2$, we can find two endomorphisms A and B commmuting with the group action on V with $AB \neq BA$. By Hahn-Banach theorem there exists a closed linear subspace $W \subset F$, such that $F = V \oplus W$. Extend A and B to F by $A|W = B|W = 0$ and define

$$\tilde{A} = \int_K \varrho(k) A \varrho(k)^{-1} \cdot d\mu(k), \quad \tilde{B} = \int_K \varrho(k) B \varrho(k)^{-1} \cdot d\mu(k).$$

Then $\tilde{A}, \tilde{B} \in \mathcal{A}(\varrho)$ and $\tilde{A}\tilde{B} \neq \tilde{B}\tilde{A}$ by construction. □

Remark 2.4. Multiplicity-free unitary representations (of arbitrary groups) are usually defined as those for which the commuting algebra is commutative. We refer the reader to [Ko], see Definition 1.5.3, where the notion is generalized to representations in topological vector spaces. In our context, i.e., for compact Lie groups acting in Fréchet spaces, this amounts to Definition 2.1.

We are mostly interested in representations arising from transformation groups of complex spaces. So let X be a reduced complex space, $\mathcal{O}(X)$ the corresponding algebra of holomorphic functions, and K a Lie group acting on X by holomorphic transformations. Then the action of K on X induces a linear representation in the Fréchet space $F = \mathcal{O}(X)$, namely,

$$\varrho(k)f(x) = f(k^{-1}x) \quad (k \in K, x \in X),$$

and we write $\mathcal{O}_\delta(X)$ for F_δ. In this setting we have $F^\infty = F$, see e.g. [A1], Sect. 5.2. Thus, if K is compact we obtain the following corollary of Theorem 2.2.

Theorem 2.5. *For any $f \in \mathcal{O}(X)$ one has*

$$f = \sum_{\delta \in \hat{K}} E_\delta f \ ,$$

where the series is compactly convergent. The series is finite if and only if f is K-finite.

Example 2.6. A Reinhardt domain $D \subset \mathbb{C}^n$ is a domain invariant under the action of $K = \mathrm{U}(1)^n$ by rotations

$$(z_1, \ldots, z_n) \to (e^{i\varphi_1} z_1, \ldots, e^{i\varphi_n} z_n).$$

The series in Theorem 2.5 is the Laurent series of $f \in \mathcal{O}(D)$. All non-zero isotypic components of $F = \mathcal{O}(D)$ are one-dimensional, and so $\mathcal{O}(D)$ is multiplicity-free.

Example 2.7. Let V be a complex vector space, $K \subset \mathrm{GL}(V)$ a connected compact linear group, and $G = K^{\mathbb{C}} \subset \mathrm{GL}(V)$ the complex reductive group containing K as a maximal compact subgroup. Assume that

($*$) each irreducible G-module occurs in $\mathbb{C}[V]$ at most once.

Then the expansion of $f \in \mathcal{O}(V)^0$ in a series of homogeneous polynomials has only finitely many non-zero terms, for otherwise the series from Theorem 2.5 would be infinite. In other words, $f \in \mathbb{C}[V]$. Moreover, an isotypic component of $\mathcal{O}(V)$ is contained in a subspace of homogeneous polynomials and is in fact irreducible. Thus ($*$) holds if and only if the representation of K in $\mathcal{O}(V)$ is multiplicity-free.

We have shown that ($*$) yields the equality $\mathcal{O}(V)^0 = \mathbb{C}[V]$. Furthermore, if $D \subset V$ is a K-invariant domain containing $0 \in V$ then ($*$) implies that $\mathcal{O}(D)^0 = \mathbb{C}[V]$. To see this, introduce a K-invariant Hermitian metric in V, take a ball in D centered at 0, restrict $f \in \mathcal{O}(D)^0$ to that ball and apply the above argument. A complete list of irreducible reductive linear groups with property ($*$) was found by V.Kac, see [K]. The result was extended by A.S.Leahy to all connected reductive linear groups, see [L].

Example 2.8. Let D be a bounded symmetric domain and let K be the isotropy subgroup of a point $o \in D$. Then the representation of K in $\mathcal{O}(D)$ is multiplicity-free.

One proof of this fact can be obtained from Theorem 4.4. Namely, the isotropy representation of K in the holomorphic tangent space $V = T_o(D)$ gives rise to a multiplicity-free

polynomial algebra $\mathbb{C}[V]$, see [J]. Of course, if we assume D irreducible then V as a $K^{\mathbb{C}}$-module occurs in the list in [K]. It is known that D can be realized as a K-invariant domain in V. Therefore the representation of K in $\mathcal{O}(D)$ is multiplicity-free by Theorem 4.4, (c) $\Rightarrow$ (b). For another proof see Example 5.4.

In the sequel, we will always assume that the compact Lie group K is connected. This is justified by the following theorem.

Theorem 2.9. *Let K be a compact Lie transformation group of an irreducible reduced complex space X. Let $K^\circ \subset K$ be the connected component of the neutral element. Then $\mathcal{O}(X)$ is multiplicity-free with respect to K if and only if $\mathcal{O}(X)$ is multiplicity-free with respect to K°.*

Proof. If $\mathcal{O}(X)$ is multiplicity-free under K° then the same is obviously true for K. To prove the converse, assume V_1, V_2 are two distinct isomorphic K°-modules in $\mathcal{O}(X)$. Let $f_1 \in V_1,\ f_2 \in V_2$ be their highest weight vectors with respect to some Borel subalgebra of the complexified Lie algebra of K°. Note that for any integer m the functions $f_1^k \cdot f_2^{m-k}$, $k = 0, \ldots, m$, are linearly independent highest weight vectors of the same weight. Thus there exists $\delta_0 \in \widehat{K^\circ}$, such that

$$\dim \mathcal{O}_{\delta_0}(X) > N \cdot d(\delta_0),$$

where $N > 0$ is an arbitrary constant. Take $N = [K : K^\circ]$ and let $\{\delta_0, \delta_1, \ldots, \delta_p\} \subset \widehat{K^\circ}$ be the set of all equivalence classes of irreducible representations obtained from δ_0 by an inner automorphism of K. Let V denote the K-submodule of $\mathcal{O}(X)$ generated by $\mathcal{O}_{\delta_0}(X)$. Then V is the direct sum of $\mathcal{O}_{\delta_i}(X),\ i = 0, 1, \ldots, p$, and all these subspaces have the same dimension. Therefore

$$\dim V = (p+1) \cdot \dim \mathcal{O}_{\delta_0}(X) > (p+1)Nd(\delta_0).$$

Let $[\delta : \delta_i]$ be the number of times δ_i occurs in the restriction of $\delta \in \hat{K}$ to K°. Frobenius reciprocity theorem shows that

$$\sum_{\delta \in \hat{K}} [\delta : \delta_i] \cdot d(\delta) = Nd(\delta_i) = Nd(\delta_0)$$

for all i, $i = 0, 1, \ldots, p$. Now, V is a multiplicity-free K-module whose irreducible components are among V_δ with $[\delta : \delta_i] \geq 1$ for some i. It follows that

$$\dim V \leq N(p+1)d(\delta_0),$$

which contradicts the above lower bound. □

3. Spherical varieties and their holomorphic counterparts

We recall the following definition from the theory of algebraic groups. All varieties and morphisms are defined over $\mathbb{C}$.

Definition 3.1. Let G be a connected reductive algebraic group acting on a normal algebraic variety Y and let $B \subset G$ be a Borel subgroup. Then Y is called spherical (with respect to G) if B has an open orbit on Y.

We refer the reader to [Lu], Sect. 0.2, for several possible answers to the question: "Pourquoi s'intéresser aux variétés sphériques?" Our motivation comes from the following result.

Theorem 3.2 (see [S], [VK]). *Let L be an algebraic line bundle on Y with G-linearization. If Y is spherical then the G-module of regular global sections $\Gamma(Y, L)$ is multiplicity-free, i.e., each irreducible G-module occurs in $\Gamma(Y, L)$ with multiplicity 0 or 1. If Y is affine and $\mathbb{C}[Y]$ is a multiplicity-free G-module then Y is a spherical G-variety. In particular, a normal affine G-variety Y is spherical if and only if the algebra $\mathbb{C}[Y]$ is a multiplicity-free G-module.*

In complex analysis, it is quite often that a real form G_0 of a complex Lie group G acts on a complex space X by holomorphic transformations, whereas G acts on X only locally. In this setting, the Lie homomorphism makes the complex Lie algebra $\mathfrak{g}$ into an algebra of holomorphic vector fields on X. A typical example is a G_0-invariant domain in a complex space acted on by G.

Definition 3.3. Let G_0 be a real form of a connected complex reductive algebraic group G. Let X be a normal (irreducible, reduced) complex space on which G_0 acts by holomorphic transformations. Then X is said to be spherical (with respect to G_0) if there exists a point $x \in X$ such that the holomorphic tangent space $T_x(X)$ is generated by vector fields from $\mathfrak{b}$, the Lie algebra of B.

Remark 3.4. If G acts holomorphically on X then this definition is independent of G_0 and amounts to saying that B has an open orbit on X. The following theorem shows that there exist non-algebraic actions of this type. In what follows, we denote by L' the commutator subgroup of a group L.

Theorem 3.5. *Let $X = G/H$ be a compact complex homogeneous space. Then X is spherical in the sense of Definition 3.3 if and only if X is a locally trivial holomorphic fiber bundle over a flag manifold with a complex torus as a fiber.*

Proof. By the normalizer theorem due to J.Tits, A.Borel and R.Remmert (see e.g. [A1], Sect. 3.5., and references therein), the normalizer of the connected component $H^\circ \subset H$ is a parabolic subgroup in G, which will be denoted by P.

Assume first that X is spherical and fix a Borel subgroup $B \subset G$ such that, on the Lie algebra level, $\mathfrak{g} = \mathfrak{b} + \mathfrak{h}$. Choose a maximal algebraic torus T in $P \cap B$ and introduce an ordering of the root system in such a way that $\mathfrak{b} = \mathfrak{b}^-$. Since H° is normalized by T, the Lie algebra $\mathfrak{h}$ is spanned by $\mathfrak{h} \cap \mathfrak{t}$ and the root vectors e_α which are contained in

$\mathfrak{h}$. The decomposition $\mathfrak{g} = \mathfrak{b}^- + \mathfrak{h}$ shows that all positive root vectors are in $\mathfrak{h}$, hence $[\mathfrak{b}^+, \mathfrak{b}^+] \subset \mathfrak{h} \subset \mathfrak{p}$. Now, if α is a positive root such that $e_{-\alpha} \in \mathfrak{p}$ then

$$[e_{-\alpha}, e_\alpha] \in [\mathfrak{p}, \mathfrak{h}] \subset \mathfrak{h}$$

and

$$e_{-\alpha} \in \mathbb{C} \cdot [e_{-\alpha}, [e_{-\alpha}, e_\alpha]] \in [\mathfrak{p}, \mathfrak{h}] \subset \mathfrak{h}.$$

Therefore $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{h}$ and $P' \subset H$, showing that the fiber P/H of the fibration $G/H \to G/P$ is a complex torus.

To prove the converse, we recall that compact complex homogeneous manifolds, fibered by tori over flag manifolds, are well-understood. In particular, the existence of a parabolic subgroup $P \subset G$ with $P' \subset H \subset P$ follows from the normalizer theorem, see [A1], Sect. 3.5, Cor. 2, and Sect. 3.6, Prop. 1. On the other hand, if P is a parabolic subgroup then G/P' is a spherical algebraic variety. Thus, for any closed complex Lie subgroup H containing P', the corresponding homogeneous space G/H is spherical in the sense of Definition 3.3. □

Corollary 3.6. *Let $X = G/H$ be a compact complex homogeneous space. Then X spherical in the sense of Definition 3.3 if and only if there exists a parabolic subgroup $P \subset G$, such that $P' \subset H \subset P$ and P/H is compact. Moreover, such an X is an algebraic homogeneous space (which is then spherical in the sense of Definition 3.1) if and only if $H = P$ and X is a flag manifold.*

Proof. The first statement follows from the proof of Theorem 3.5. The second statement is an extract from the theory of algebraic groups. Namely, if $X = G/H$ is an algebraic homogeneous space then H is an algebraic subgroup of G. By Chevalley's theorem X is a quasi-projective variety with an equivariant projective embedding. Since X is also compact, the image of this embedding is closed and Borel's fixed point theorem shows that H is parabolic. □

4. Spherical Stein spaces

In this section, we carry over Theorem 3.2 to the category of complex spaces.

Lemma 4.1 (see [AH], Lemma 2). *Let $\Omega \subset \mathbb{C}^n$ be a domain containing the origin and let*

$$A_i = \sum_{j=1}^{n} a_{ij}(z) \frac{\partial}{\partial z_j}$$

be holomorphic vector fields in Ω, such that $a_{ij}(0) = \delta_{ij}$. If $f \in \mathcal{O}(\Omega)$, $A_i f \in \mathcal{O}(\Omega) f$, and $f(0) = 0$ then $f = 0$.

Theorem 4.2 (cf. [AH], Theorem 1). *We use the notations from Definition 3.3. Let X be a spherical complex space with respect to G_0 and let L be a holomorphic line bundle*

on X with G_0-linearization. Then each irreducible finite-dimensional G_0-module occurs in $\Gamma(X,L)$ with multiplicity 0 or 1.

Proof. We can find a non-singular point $x \in X$, a coordinate neighborhood Ω of $x \in X$ with coordinates $z_i,\ z_i(x) = 0$, and the vector fields A_i in Ω coming from $\mathfrak{b}$ and satisfying the assumptions of Lemma 4.1. Let V_1 and V_2 be two distinct, but isomorphic irreducible finite-dimensional G_0-submodules in $\Gamma(X,L)$. Note that V_1 and V_2 are also irreducible and isomorphic $\mathfrak{g}$-modules. Thus they have the same highest weight $\lambda : \mathfrak{b} \to \mathbb{C}$, and so we obtain two linearly independent holomorphic global sections $s_i \in \Gamma(X,L)$, such that

$$As_i = \lambda(A)s_i \quad (A \in \mathfrak{b},\ i = 1,2).$$

Let s be a non-zero linear combination of s_1, s_2 vanishing at x. Shrinking Ω, we can find a section $s_0 \in \Gamma(\Omega, L)$ without zeros. Define $\varphi_A \in \mathcal{O}(\Omega)$ by $As_0 = \varphi_A s_0$ and write $s|\Omega = fs_0$ for some $f \in \mathcal{O}(\Omega)$. Then

$$Af = (\lambda(A) - \varphi_A)f \in \mathcal{O}(\Omega)f$$

by Leibniz rule. Therefore $f = 0$ by Lemma 4.1. Since $s \neq 0$, we get a contradiction. $\square$

Theorem 4.3. *Let Y be a normal Stein space with holomorphic action of G and let X be a G_0-invariant domain in Y. The following conditions are equivalent:*

(a) *X is spherical with respect to G_0;*

(b) *any irreducible finite-dimensional G_0-module occurs in $\mathcal{O}(X)$ with multiplicity 0 or 1;*

(c) *Y is a spherical affine variety of G.*

Proof. (a) $\Rightarrow$ (b) by Theorem 4.2 and (c) $\Rightarrow$ (a) is evident. To prove (b) $\Rightarrow$ (c), observe that the trivial G-module occurs in $\mathcal{O}(Y)$ only once. For, if $f_1, f_2 \in \mathcal{O}(Y)$ were G-invariant and non-proportional then $f_1|X, f_2|X$ were G_0-invariant and also non-proportional, contradictory to (b). Since two closed G-orbits on Y are separated by G-invariant holomorphic functions, there is only one such orbit. It follows that Y is (G-equivariantly biholomorphic to) an affine algebraic variety on which G acts algebraically, see [Sn], Cor. 5.6. Furthermore, $\mathbb{C}[Y]$ is a multiplicity-free G-module. Indeed, two distinct isomorphic irreducible G-submodules of $\mathbb{C}[Y]$ would induce by restriction to X two distinct isomorphic irreducible G_0-submodules of $\mathcal{O}(X)$, which again contradicts (b). By Theorem 3.2 the variety Y is spherical. $\square$

For G_0 compact one has a stronger result.

Theorem 4.4 (see [AH], Theorem 2). *Let K be a compact real form of G and let X be a normal Stein K-space. Then the following conditions are equivalent:*

(a) *X is spherical with respect to K;*

(b) *$\mathcal{O}(X)$ is a multiplicity-free K-module;*

(c) *X is a K-invariant domain in a spherical affine G-variety.*

Proof. Again, in view of Theorem 4.2 we only have to prove (b) $\Rightarrow$ (c). Now, for compact transformation groups one has the so called complexification theorem, see [He]. Namely, there exists another reduced Stein space $X^{\mathbb{C}}$ acted on by G along with a K-equivariant open embedding $i : X \hookrightarrow X^{\mathbb{C}}$. Moreover, $G \cdot i(X) = X^{\mathbb{C}}$, so $X^{\mathbb{C}}$ is normal if X is normal. To finish the proof, it remains to take $Y = X^{\mathbb{C}}$ in Theorem 4.3. □

5. Antiholomorphic involutions

For complex manifolds, J.Faraut and E.G.F.Thomas gave an interesting and simple geometric condition which implies that the function algebra is a multiplicity-free module, see [FT]. In their setting, the group of holomorphic transformations acting on the manifold X is not necessarily compact and the condition guarantees that any invariant Hilbert subspace in $\mathcal{O}(X)$ is multiplicity-free (see Remark 2.4). We state their result for compact Lie groups acting on complex spaces.

Definition 5.1. An antiholomorphic self-map μ of a complex space X is called an antiholomorphic involution if $\mu^2 = \mathrm{id}$.

Theorem 5.2 (see [FT], Theorem 3). *Let X be an irreducible reduced complex space and K a compact Lie group acting on X by holomorphic transformations. Assume there exists an antiholomorphic involution $\mu : X \to X$ such that $\mu(x) \in K \cdot x$ for every $x \in X$. Then $\mathcal{O}(X)$ is a multiplicity-free K-module.*

Remark 5.3. The proof in [FT] goes without changes for irreducible reduced complex spaces. A simplified proof is given in [AP], see Proposition 3.3.

Example 5.4. Let D be a bounded symmetric domain in a complex vector space V. Further, let $\mathfrak{g}$ be the Lie algebra of the automorphism group $\mathrm{Aut}(D)$ and let $\mathfrak{k}$ be the subalgebra of $\mathfrak{g}$ associated to a maximal compact subgroup $K \subset \mathrm{Aut}(D)$. We assume that $0 \in D$ and K is the isotropy subgroup of 0. In the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, we can identify $\mathfrak{p}$ with V (as a real vector space) and consider a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$ as a real vector subspace of V.

For every bounded symmetric domain there exists an antilinear map $\mu : V \to V$, such that $\mu^2 = \mathrm{id}$, $\mu(D) = D$ and $\mu(z) = z$ for all z in some Cartan subspace $\mathfrak{a}$. For irreducible bounded symmetric domains this fact is checked using the classification (see [FT]), and the general case follows easily.

Now, any $x \in D$ can be written as $x = k \cdot z$ with $k \in K$, $z \in \mathfrak{a} \cap D$, and so we obtain $\mu(x) = \mu(k \cdot z) = \mu k \mu(z) \in K \cdot x$. Thus, Theorem 5.2 implies that $\mathcal{O}(D)$ is a multiplicity-free K-module.

6. Spherical Stein manifolds and the Weyl involution

In the final section, we discuss the converse to Theorem 5.2. For this, additional assumptions on X are necessary. Namely, X must have sufficiently many holomorphic functions.

Indeed, if e.g. $\mathcal{O}(X) = \mathbb{C}$ and K is trivial then $\mathcal{O}(X)$ is multiplicity-free, but a self-map of X preserving K-orbits is the identity map which is holomorphic and not antiholomorphic. In the end, we will assume that X is a Stein manifold, but our first result holds in a more general setting. To state this result, we need several definitions.

Definition 6.1. Given a group Γ acting on two sets A and B and an automorphism $\vartheta : \Gamma \to \Gamma$, one says that a map $\mu : A \to B$ is ϑ-equivariant if $\mu(\gamma \cdot a) = \vartheta(\gamma) \cdot \mu(a)$ for all $a \in A$, $\gamma \in \Gamma$.

Definition 6.2. An involutive automorphism θ of a connected compact Lie group K is called a Weyl involution if $\theta(t) = t^{-1}$ for all t in a maximal torus $T \subset K$.

Remark 6.3. It is known that such an involution exists and that two Weyl involutions are conjugate by an inner automorphism of K, see e.g. [W], Sect. 12.6. From the point of view of the representation theory, the main property of θ is the following one. For a representation $\varrho : K \to \mathrm{GL}(V)$, the representation $k \mapsto \varrho(\theta(k))$ in the same vector space V is dual to the given one. The notion of a Weyl involution θ can also be defined for a connected reductive algebraic group G over $\mathbb{C}$. The definition and the properties of $\theta : G \to G$ are similar.

Theorem 6.4 (see [AP], Theorem 4.1). *Let X be a holomorphically separable irreducible reduced complex space, K a connected compact Lie group acting on X by holomorphic transformations, $\theta : K \to K$ a Weyl involution, and $\mu : X \to X$ a θ-equivariant anti-holomorphic involution of X. Then $\mathcal{O}(X)$ is a multiplicity-free K-module if and only if $\mu(x) \in K \cdot x$ for every $x \in X$.*

Proof. Without going into details, we explain the idea, showing the role of the Weyl involution. Let $V \subset \mathcal{O}(X)$ be an irreducible K-submodule. Introduce a K-invariant Hermitian inner product and choose a unitary basis $\{f_j\}$ in V. It is easily seen that the function

$$\Phi = \sum_j f_j \bar{f}_j$$

is K-invariant and does not depend on the choice of basis. Assuming $\mathcal{O}(X)$ multiplicity-free, we get a family of real-analytic functions $\{\Phi_\delta\}$, one for each isotypic component $\mathcal{O}_\delta(X)$. Using the fact that X is holomorphically separable, one can prove that these functions separate K-orbits.

On the other hand, let $\mu f(x) = f(\mu(x))$ for any function f on X. Since μ is θ-equivariant, $\mu\mathcal{O}_\delta(X)$ is an irreducible K-module with representation twisted by θ. But θ is the Weyl involution, so this K-module is dual to $\mathcal{O}_\delta(X)$. Since the K-module $\overline{\mathcal{O}_\delta(X)}$ is also dual to $\mathcal{O}_\delta(X)$ and $\mathcal{O}(X)$ is multiplicity-free, we have the equality

$$\mu\mathcal{O}_\delta(X) = \overline{\mathcal{O}_\delta(X)}.$$

Moreover, one can show that the composition of μ with complex conjugation preserves a K-invariant Hermitian inner product in $\mathcal{O}_\delta(X)$. From this it follows that $\mu\Phi_\delta = \Phi_\delta$. Since

the family $\{\Phi_\delta\}$ separates K-orbits, μ must preserve each of them, i.e., $\mu(x) \in K \cdot x$ for all $x \in X$.

The converse is already known from Theorem 5.2. $\square$

Lemma 6.5 (see [A2], Theorem 3.1). *Let L be a connected compact Lie group, $\theta : L \to L$ a Weyl involution, and $\varrho : L \to \mathrm{GL}(V)$ a complex representation of L. Then there exists an antilinear map $\nu : V \to V$, such that $\nu^2 = \mathrm{id}$ and*

$$\nu(\varrho(l)v) = \varrho(\theta(l))\,\nu(v)$$

for all $l \in L$, $v \in V$. If ϱ is irreducible and $\nu' : V \to V$ is another map with the same properties then $\nu' = c\nu$ for some c with $|c| = 1$.

Definition 6.6 (see [AV]). Let G be a connected complex reductive group and $H \subset G$ a reductive algebraic subgroup. Then H is called adapted if there is a Weyl involution $\theta : G \to G$, such that $\theta(H) = H$ and θ induces a Weyl involution of the connected component H°.

Theorem 6.7. *Let $H \subset G$ be connected and adapted and let $X = G \times_H V$ be a homogeneous fiber bundle with H acting on V by a regular linear representation. For any maximal compact subgroup $K \subset G$ and any Weyl involution $\theta : K \to K$ there exists a θ-equivariant antiholomorphic involution $\mu : X \to X$.*

Proof. It suffices to prove the theorem for some K and θ. Let L be a maximal compact subgroup of H. Note that L is connected. Choose a maximal compact subgroup $K \subset G$ containing L. According to [AV], Prop. 5.14, we can find a Weyl involution θ of G so that K and L are θ-invariant and the restriction of θ to K and to L is a Weyl involution of these groups. Moreover, if $\tau : G \to G$ is the Cartan involution with fixed point subgroup K then $\tau\theta = \theta\tau$. The product $\sigma = \tau\theta$ is an antiholomoprphic involution of G. We remark that the fixed point subgroup of σ is a split real form of G.

We can identify H and L with their images in $\mathrm{GL}(V)$ and assume that ϱ in Lemma 6.5 is the identity representation. By that lemma there exists an antilinear involution $\nu : V \to V$, such that

$$\nu(lv) = \theta(l)\nu(v)$$

for all $l \in L$, $v \in V$. We claim that in fact

$$\nu(hv) = \sigma(h)\nu(v)$$

for all $h \in H$. Indeed, if $v \in V$ is fixed then the second equality holds for all h in a maximal totally real submanifold $L \subset H$ (where it is just the first one) and therefore everywhere on H.

Consider the antiholomorphic involution $\tilde{\mu}$ of $G \times V$, defined by

$$\tilde{\mu}(g, v) = (\sigma(g), \nu(v)).$$

For $h \in H$ let $t_h(g, v) = (gh^{-1}, hv)$. The transformations t_h define an action of H on $G \times V$, and X is the geometric quotient of that action. Since $\nu(hv) = \sigma(h)\nu(v)$, it follows that $\tilde{\mu} \cdot t_h = t_{\sigma(h)} \cdot \tilde{\mu}$. Thus $\tilde{\mu}$ gives rise to a self-map $\mu : X \to X$. Clearly, μ is an antiholomorphic involution. Since θ and σ coincide on K, the map μ is θ-equivariant with respect to the K-action on X. $\square$

Lemma 6.8 (see [KVS], Cor. 2.2). *Let G be a connected complex reductive group and let X be a smooth affine spherical variety of G. Then $X = G \times_H V$, where $H \subset G$ is a reductive subgroup with G/H spherical and V is a spherical H-module.*

Theorem 6.9 (see [A2], Theorem 1.2). *Let X be a Stein manifold acted on by a connected compact Lie group K of holomorphic transformations. Let $\theta : K \to K$ be a Weyl involution. If X is spherical with respect to K then there exists a θ-equivariant antiholomorphic involution $\mu : X \to X$. Any such involution preserves K-orbits.*

Proof. We write G for the complexified group $K^{\mathbb{C}}$. As in the proof of Theorem 4.4, we can reduce the statement to the algebraic case. Namely, X can be embedded as a K-invariant domain into an affine spherical G-variety $X^{\mathbb{C}}$. Moreover, the G-saturation of X in $X^{\mathbb{C}}$ is the whole $X^{\mathbb{C}}$, and so $X^{\mathbb{C}}$ is non-singular. Assume that the theorem is proved for $X^{\mathbb{C}}$. Then we have an antiholomorphic involution $\mu : X^{\mathbb{C}} \to X^{\mathbb{C}}$, which is θ-equivariant with respect to the K-action. By Theorem 6.4 μ preserves K-orbits on $X^{\mathbb{C}}$. Thus the subset $X \subset X^{\mathbb{C}}$ is K-stable. The restriction of μ to X is the involution we are looking for, and the second assertion is now obvious.

Let now X be a smooth affine spherical variety. Then Lemma 6.8 displays X as a vector bundle, $X = G \times_H V$. Moreover, since G/H is a spherical variety, the subgroup H is adapted, see [AV], Prop. 5.10. If H is connected then we can apply Theorem 6.7 and finish the proof. For the general case see [A2].

Corollary 6.10. *Let X, K and θ be as in Theorem 6.9. The following properties of the action $K \times X \to X$ are equivalent:*

(a) *$\mathcal{O}(X)$ is a multiplicity-free K-module;*

(b) *X is a spherical K-manifold;*

(c) *there exists an antiholomorphic involution $\mu : X \to X$ preserving K-orbits;*

(d) *there exists a θ-equivariant antiholomorphic involution $\mu : X \to X$ preserving K-orbits.*

Proof. This results from Theorems 4.4, 5.2, and 6.9. $\square$

Remark 6.11. The assumption that X is non-singular is only important in the proof of (b) $\Rightarrow$ (d). The author does not know whether this is true for spaces with (normal) singularities.

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